Note on scale invariance and self-similar evolution in (3+1)-dimensional signum-Gordon model

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Abstract

Several classes of self-similar, spherically symmetric solutions of relativistic wave equation with a nonlinear term of the form $\mathrm{sign}(\varphi)$ are presented. They are constructed from cubic polynomials in the scale invariant variable t/r. One class of solutions describes a process of wiping out the initial field, another an accumulation of field energy in a finite and growing region of space.

PACS: 11.27.+d, 98.80.Cq, 11.10.Lm

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1 Introduction

Symmetry transformations for a given equation act within the space of its solutions. Especially interesting are fixed points of such transformations, i.e., the solutions that remain invariant. In the case of continuous symmetries they depend on a reduced number of independent variables. This significantly simplifies the task of finding the solutions, the task that so often is formidably difficult in the case of nonlinear field equations. Apart from the rudimentary examples of rotational and translational symmetries, in certain models there is also a one-parameter scaling symmetry. The corresponding invariant solutions, called the self-similar ones, play an important role in mathematical analysis of nonlinear evolution equations, as well as in physics oriented investigations, see, e.g., the books [1], [2] and a sample of recent research papers [3].

The present paper is a sequel to papers [4], where self-similar solutions in the signum-Gordon model with a real scalar field in 1+1 dimensions were analyzed. Rich variety of such solutions was found. Certain related models were investigated in [5]. Derivation and a general discussion of the signum-Gordon model can be found in [4]. The model is very interesting, there is no doubt that its investigations should be continued in several directions, such as finding classical solutions of various kinds, applications, or quantization.

As is well-known, field theories in 1+1 dimensions have rather special properties. Therefore, it is not clear whether the amazing picture of the space of self-similar solutions found in [4] is valid also in higher dimensions. This is our main motivation for the present work. Another attraction is the possibility of obtaining several exact analytic solutions to the signum-Gordon equation in 3+1 dimensions, and to have new insights into rather intriguing dynamics of the scalar field in that model.

We have found several families of self-similar, spherically symmetric analytic solutions. They are composed from cubic polynomials in the scale invariant variable $u = t/r$. Our results show that there is a qualitative similarity between the 3+1 and 1+1 dimensional cases, but significant differences appear. First, the lack of translational invariance in the radial variable r results in the absence of counterparts of certain classes of 1+1 dimensional solutions. Second difference, perhaps not so unexpected, is that the pertinent calculations are a bit more complicated because we have to look for zeros of cubic polynomials, while in the 1+1 dimensional case only the second order polynomials were present. The exact solutions we have found provide examples of non-trivial evolution of the scalar field φ. Especially interesting seem to be solutions of the types II and III, presented in

Section 3, which show how the field settles exactly at the vacuum value $\varphi = 0$.

The plan of our paper is as follows. In Section 2 we describe the method of constructing the self-similar solutions in the signum-Gordon model. Explicit solutions in the 3+1 dimensional case are presented in Section 3. Section 4 is devoted to discussion of our results.

2 Scale invariant Ansatz and the method of constructing the solutions

The signum-Gordon equation for the real scalar field φ in $D + 1$ dimensional space-time reads

$$\partial_\mu \partial^\mu \varphi(x) + \text{sign}(\varphi(x)) = 0. \tag{1}$$

The sign function takes the values $\pm 1, 0$, $\text{sign}(0) = 0$. It is clear that Eq. (1) implies that the second derivatives of φ may not exist if φ vanishes at isolated points, but even at such points one-sided second order derivatives do exist. The proper mathematical framework for discussing equations of such kind is well-known: one should consider the so called weak solutions, see, e.g., [6]. All solutions constructed below have been checked in this respect.

From a given solution $\varphi(x)$ of Eq. (1) one can obtain one-parameter family of solutions of the form

$$\varphi_\lambda(x) = \lambda^2 \varphi(\frac{x}{\lambda}), \tag{2}$$

where $\lambda > 0$ is a constant. The solution is self-similar if $\varphi_\lambda(x) = \varphi(x)$ for all $\lambda > 0$. Taking $\lambda = r$, where r is the radius, and assuming the spherical symmetry, we may then write

$$\varphi(x) = r^2\, G(\frac{t}{r}), \tag{3}$$

where $t = x^0$ is the time. With this Ansatz, Eq. (1) gives

$$(u^2 - 1)G'' - (D + 1)uG' + 2DG - \text{sign}(G) = 0, \tag{4}$$

where $u = t/r$ and $G' = dG/du$. This equation has the particular solutions

$$G = 0, \quad G = \pm \frac{1}{2D},$$

which are trivial, but nevertheless play important role below.

Let $g(u)$ be a solution of the auxiliary linear (!) equation

$$(u^2-1)g''-(D+1)ug'+2Dg=0. \tag{5}$$

Then $G_+(u)=g(u)+1/(2D)$ is a solution of (4) on the interval of u defined by the condition $G_+(u)>0$, and $G_-(u)=g(u)-1/(2D)$ on the interval in which $G_-(u)<0$. It turns out that patching together a number of such partial solutions and, in some cases, including also the trivial solution $G=0$, one can cover the whole interval $0\le u<\infty$. In this way we obtain a self-similar solution of Eq. (1) valid for all $r\in[0,\infty)$ and $t\in[0,\infty)$. The values $G(0),G'(0)$ determine the initial data for $\varphi(t,r)$:

$$\varphi(0,r)=r^2G(0),\;\;(\partial_t\varphi)(0,r)=rG'(0). \tag{6}$$

When patching the partial solutions we demand continuity of $G(u)$, and also continuity of $G'(u)$, unless $u=1$. $G'(u)$ at $u=1$ does not have to be continuous because G'' in Eq. (1) is multiplied by the factor u^2-1 which vanishes at that point. This is not surprising because $u=1$ corresponds to the light-cone $r=t$, the characteristic hypersurface for equation (1).

Equation (5) has two linearly independent solutions:

$$g_1(u)=Du^2+1,\quad g_2(u)=\sum_{l=0}^{\infty}c_l u^{2l+1}, \tag{7}$$

where the coefficients c_l are determined from the recurrence relation

$$c_{l+1}=\frac{(2l-1)(2l+1-D)}{2(l+1)(2l+3)}c_l. \tag{8}$$

Note that in the case of odd space dimension D the series for g_2 is in fact a polynomial of order D, because $c_l=0$ for all $l>(D-1)/2$. In particular, $g_2=u$ for $D=1$, $g_2(u)=u^3+3u$ when $D=3$, and $g_2=u^5-10u^3-15u$ for $D=5$, apart from overall multiplicative constants.

In the next Section we consider in detail the $D=3$ case.

3 Explicit self-similar solutions in the case $D=3$

In the case of three space dimensions the general form of the partial solutions reads

$$G=\pm\frac{1}{6}+\alpha(3u^2+1)+\beta(u^3+3u), \tag{9}$$

or equivalently,

$$G = \pm\frac{1}{6} + \gamma(u-1)^3 + \delta(u+1)^3, \tag{10}$$

where $\alpha, \beta, \gamma, \delta$ are constants. The sign in front of 1/6 has to be equal to sign(G).

Let us begin our exploration of the space of the self-similar solutions by checking their asymptotic behavior at $u \to \infty$. It turns out that G has to have a constant sign if values of u are large enough – we prove in the Appendix the following lemma.

Lemma. Suppose that $u_1 > 1$ is a real zero of the polynomial (9). Then, that polynomial does not have any real zeros larger than u_1.

It follows that in the region $u \geq u_1$ the solution has the fixed form (9) with certain fixed coefficients α, β. The just excluded option was that of having an infinite sequence of polynomials with alternating signs. Now, let us recall that the region $u \to \infty$ corresponds to $r \to 0$ ($t > 0$). Therefore, the polynomial $u^3 + 3u$ would give a singular field $\varphi = r^2 G \sim 1/r$ with a divergent, non integrable at $r = 0$ energy density. For this reason, we put $\beta = 0$. Note also that by taking into account the symmetry $\varphi \to -\varphi$, we may consider just two cases of the asymptotic form of G at large values of u: $G(u) > 0$, or $G(u) = 0$. The latter case is considered in the second and third subsections (solutions of the types II and III). The former case is named type I.

3.1 Solutions of type I

In this case, for arbitrary large values of u

$$G(u) = \frac{1}{6} + \alpha(3u^2 + 1) > 0,$$

as pointed out in the preceding paragraph. This implies that $\alpha \geq 0$, and then such $G(u)$ does not have any zeroes down to $u = 1$, where it can be glued with another polynomial of the form (9) or (10). We conclude that all type I solutions on the interval $u \in [1, \infty)$ have the form

$$G_\infty(u) = \frac{1}{6} + \alpha_\infty(3u^2 + 1) \tag{11}$$

with an arbitrary constant $\alpha_\infty \geq 0$.

At the point $u = 1$ the solution G_∞ can be glued with another polynomial (9) or (10), taken with the $+1/6$ because $G_\infty(1) = 1/6 + 4\alpha_\infty > 0$. Let us denote

such polynomial by G_+ and its constants by γ_+, δ_+ (here we prefer the form (10)). The gluing condition at $u = 1$, $G_\infty(1) = G_+(1)$ gives $\delta_+ = \alpha_\infty/2$. Hence,

$$G_+(u) = \frac{1}{6} + \gamma_+(u-1)^3 + \frac{1}{2}\alpha_\infty(u+1)^3. \tag{12}$$

Now we have to determine the lower end of the interval on which G_+ is the solution (the upper end is $u = 1$). The first possibility, denoted as Ia, is that $G_+ > 0$ for all $u \in [0, 1)$. This occurs if $\gamma_+ < 1/6 + \alpha_\infty/2$. In this case G_+ and G_∞ cover the whole interval $[0, \infty)$, and these functions together give the complete solution. It has the shape sketched in Fig. 1 with the dashed line.

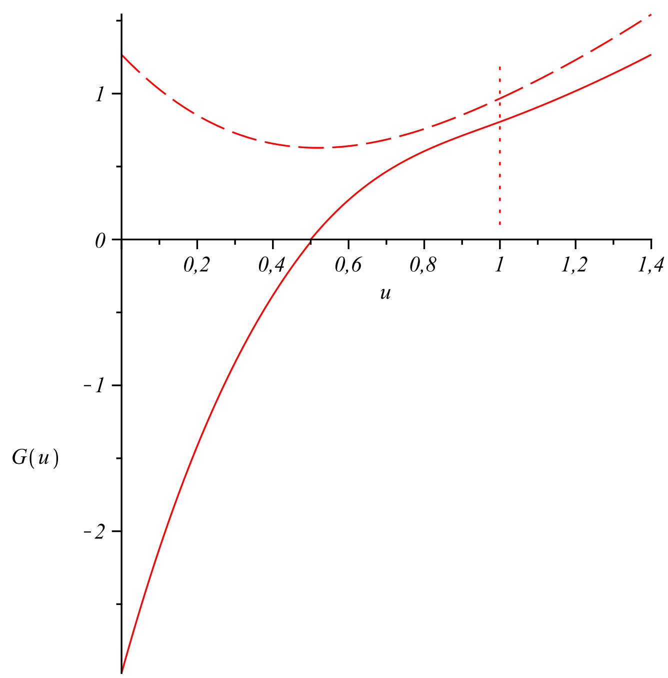

Figure 1: Solutions of the type Ia (the dashed line) and Ib (the solid line)

The other possibility, denoted as Ib, is that there exists $u_0 \in (0, 1)$ at which $G_+(u_0) = 0$. Simple calculation gives in this case

$$\gamma_+ = \frac{1 + 3\alpha_\infty(1+u_0)^3}{6(1-u_0)^3}. \tag{13}$$

Moreover, it turns out that u_0 has to be the first order zero of G_+. Therefore, at the point u_0 we may glue G_+ with a negative polynomial

$$G_- = -\frac{1}{6} + \gamma_-(u-1)^3 + \delta_-(u+1)^3 \leq 0, \tag{14}$$

and not with the trivial solution $G = 0$. The matching conditions

$$G_-(u_0) = 0, \quad G'_-(u_0) = G'_+(u_0)$$

give

$$\gamma_- = \frac{u_0 + 3\alpha_\infty(1+u_0)^3}{6(1-u_0)^3}, \;\; \delta_- = \frac{\alpha_\infty}{2} + \frac{1}{6(1+u_0)^2}. \tag{15}$$

It turns out that $G_-(u)$ found above does not have any zeros in the interval $[0, u_0)$. Therefore, the three functions: G_∞ for $u \geq 1$, G_+ for $u_0 \leq u \leq 1$, and G_- for $0 \leq u \leq u_0$, form the complete solution of Eq. (4). It is sketched in Fig. 1 with the solid line.

The values of $G(0), G'(0)$, which specify the initial values of the scalar field through formula (6), read as follows.
Type Ia:

$$G_+(0) = \frac{1}{6} - \gamma_+ + \frac{\alpha_\infty}{2}, \;\; G'_-(0) = 3\gamma_+ + \frac{3\alpha_\infty}{2}.$$

Type Ib:

$$G_-(0) = -\frac{1}{6} - \gamma_- + \delta_-, \;\; G'_-(0) = 3(\gamma_- + \delta_-).$$

Varying α_∞ in the interval $[0, \infty)$ and u_0 in the interval $[0, 1)$ we obtain certain sets in the $(G(0), G'(0))$ plane. They are shown in Fig. 2. We have denoted them Ia, Ib, identically as the corresponding solutions.

The regions $-Ia, -Ib$ are obtained by the reflection in the origin, $(G(0), G'(0)) \to (-G(0), -G'(0))$, related to the symmetry $\varphi \to -\varphi$. The half-infinite curve that separates the region Ib from IIb is obtained for $\alpha_\infty = 0$. It has the following parametric form with $u_0 \in [0, 1)$ as the parameter:

$$G(0) = \frac{1}{6(1+u_0)^2} - \frac{u_0}{6(1-u_0)^3} - \frac{1}{6}, \quad G'(0) = \frac{1}{2(1+u_0)^2} + \frac{u_0}{2(1-u_0)^3}.$$

The curve starts at the point $(0, 1/2)$ that corresponds to $u_0 = 0$, and it approaches its asymptote, that is the straight half-line

$$G(0)(\sigma) = -\frac{1}{8} - \sigma, \quad G'(0)(\sigma) = \frac{1}{8} + 3\sigma, \quad \sigma \in [0, \infty),$$

when $u_0 \to 1$.

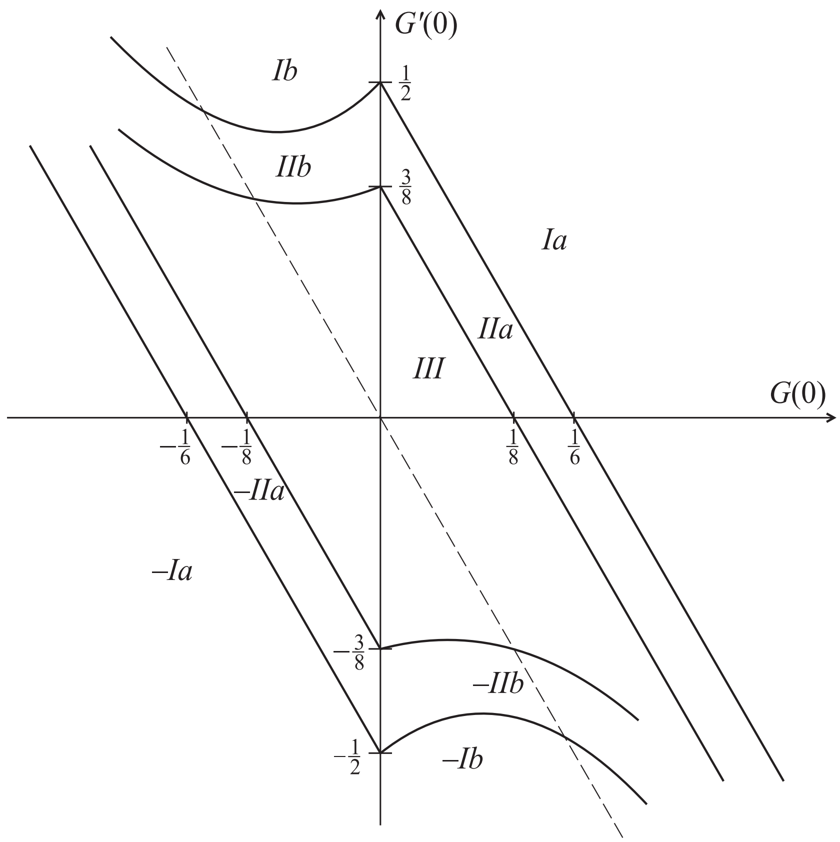

Figure 2: The map of the self-similar solutions. The regions Ia, IIa are separated from Ib, IIb, respectively, by the $G'(0)$ axis. The infinite region III extends over all area encompassed by the stripes $IIa, IIb, -IIa, -IIb$, except for the origin $(0, 0)$ which corresponds to the trivial solution $G = 0$. The stripe IIb asymptotically approaches the stripe $-IIa$, similarly IIa approaches $-IIb$. The meaning of the dashed straight line is explained in Section 4

3.2 Solutions of type II

Solutions of the type II are obtained by taking the trivial solution $G(u) = 0$ for $u \geq u_1 > 1$, and gluing it at the point u_1 with the polynomial (10) taken with the $+$ sign (the other sign can be obtained by the symmetry transformation $\varphi \to -\varphi$). Because $u_1 \neq 1$, the matching condition includes the first derivative,

$$G(u_1) = 0, \;\; G'(u_1) = 0.$$

Solving these conditions we obtain the following polynomial

$$G_d(u) = \frac{(u_1 - u)^2(2u_1 u + u_1^2 - 3)}{6(u_1^2 - 1)^2}. \tag{16}$$

It has strictly positive values in the whole interval $u \in [1, u_1)$. At $u = 1$ that polynomial is glued with another positive polynomial $G_+(u)$ of the form (9). The matching condition $G_d(1) = G_+(1)$ gives

$$G_+(u) = \frac{1}{6} - \alpha_+(u-1)^3 - \frac{u(u^2+3)}{6(1+u_1)^2}, \tag{17}$$

where α_+ is an arbitrary constant. Similarly as for the type I solutions, there are two possibilities. The first, denoted as IIa, with $G_+(u)$ strictly positive on the whole interval $(0, 1]$, occurs when $\alpha_+ \geq -1/6$. In this case we already have the complete solution: $G = 0$ for $u \geq u_1$, G_d for $u \in [u_1, 1]$, and G_+ given by formula (17) for $u \in [0, 1]$. It is sketched in Fig. 3 with the dashed line. The values of G_+, G'_+ at $u = 0$, i.e.,

$$G_+(0) = \frac{1}{6} + \alpha_+, \;\; G'_+(0) = -3\alpha_+ - \frac{1}{2(1+u_1)^2},$$

where $\alpha_+ \in [-1/6, \infty), \; u_1 \in (1, \infty)$, fill a semi-infinite straight-linear stripe in the $(G(0), G'(0))$ plane. It is denoted as IIa in Fig. 2.

The second possibility is that $G_+(u)$ has a zero at some $u_0 \in (0, 1)$. If this is the case, then

$$\alpha_+ = \frac{u_0(u_0^2+3)}{6(1-u_0)^3(1+u_1)^2} - \frac{1}{6(1-u_0)^3}.$$

Formula (17) implies that the zero can only be of the first order. The matching conditions at u_0 determine the two coefficients in the negative polynomial

$$G_-(u) = -\frac{1}{6} + \gamma_-(u-1)^3 + \delta_-(1+u)^3 \tag{18}$$

which is glued at that point with $G_+(u)$. Simple calculations give

$$\gamma_- = \frac{u_0}{6(1-u_0)^3} - \frac{(1+u_0)^3}{12(1+u_1)^2(1-u_0)^3}, \quad \delta_- = \frac{1}{6(1+u_0)^2} - \frac{1}{12(1+u_1)^2}.$$

It turns out that $G_-(u)$ is strictly negative in the whole interval $u \in [0, u_0)$ for all choices of $u_0 \in (0,1),\ u_1 \in (1,\infty)$. Thus, we have obtained again the complete solution: $G = 0$ for $u \geq u_1$, G_d for $u \in [1, u_1]$, G_+ for $u \in [u_0, 1]$, and G_- for $u \in [0, u_1]$. We denote it as IIb. It is sketched in Fig. 3 with the solid line.

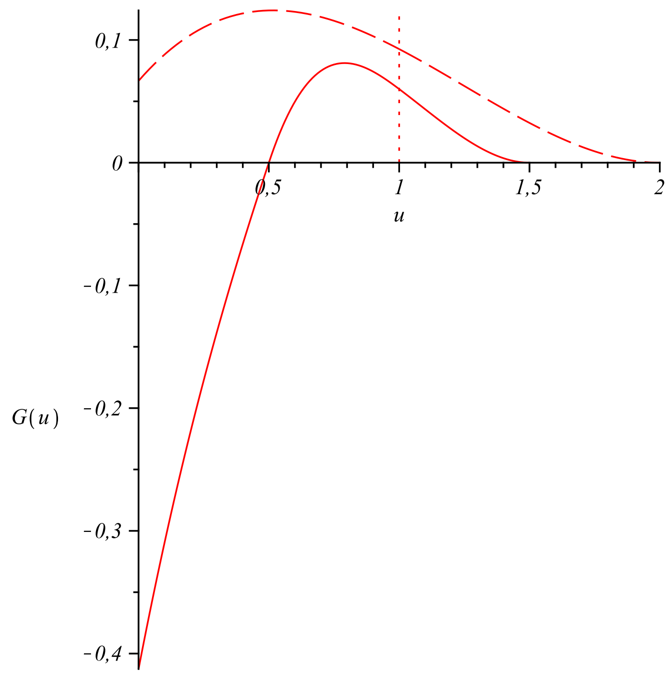

Figure 3: Solutions of the type IIa (the dashed line) and IIb (the solid line)

The points $(G_-(0), G'_-(0))$ with $u_0 \in (0,1), u_1 > 1$ fill in the $(G(0), G'(0))$ plane the region denoted as IIb, see Fig. 2. It borders the region IIa along the segment $(3/8, 1/2)$ of the $G'(0)$ axis. The region IIb is bounded from below by the curve obtained by putting $u_1 = 1$ in formulas for $G_-(0), G'_-(0)$ for the solution (18). The curve is obtained in the parametric form, namely

$$G_-(0) = -\frac{u_0(2+u_0)}{6(1+u_0)^2} - \frac{u_0(1+u_0)}{24(1-u_0)^2}, \ G'_-(0) = \frac{1}{2(1+u_0)^2} + \frac{3u_0 - 1}{8(1-u_0)^2},$$

where u_0 varies in the interval $(0, 1)$. The asymptote of this curve, approached when $u_0 \to 1-$, coincides with the upper boundary of the region $-IIa$ in Fig. 2 (the semi-infinite straight line that starts from the point $(0, -3/8)$ on the $G'(0)$ axis).

3.3 Solutions of type III

Thus far we have not found solutions for which the points $(G(0), G'(0))$ would lie in the central part of the map presented in Fig. 2. In fact, we have not exhausted yet all possibilities for having the solutions of the type II because we have assumed that $u_1 > 1$. Simple calculations show that the partial solution of the form G_d, formula (16), does not exist if $u_1 < 1$. Thus, we are left with the last possible choice, namely $u_1 = 1$. In this case $G(u) = 0$ for $u \geq 1$. The matching condition with the trivial solution $G = 0$ at $u = 1$ does not involve the first derivative. For his reason, we regard such solutions as essentially different from the ones already constructed, and we classify them as the type III.

As for the general form of such solutions we are guided by the results obtained for the signum-Gordon model in 1+1 dimensions, [4]. Thus, in the present case we expect an infinite sequence $\{G_1, G_2, G_3, \ldots\}$ of polynomials of the form (10) with alternating signs, glued together at the points $u_0, u_1, u_2, \ldots$ lying in the interval $(0, 1)$ and converging at $u = 1$. These points are the first order zeros of the pertinent polynomials, and the matching conditions include the first derivatives. Unfortunately, we have not been able to fully construct these solutions in exact, analytic form because the cubic polynomials pose significant problems in explicit calculations. One can relatively easily construct several first polynomials with the desired properties, but the infinite sequence of cubic polynomials is a different matter.

Numerical calculations support our expectations. Example of such a numerical solution is shown in Fig. 4. It has been calculated up to the point $u = 0.9996$, determined by numerical accuracy of the computation. Of course, the numerical results for this kind of a problem, where we need to check an infinite sequence of polynomials, are not decisive, even if they are quite suggestive. For this reason, the given above description of the solutions of the type III has, strictly speaking, the status of a conjecture, as opposed to the status of the solutions of the types I and II.

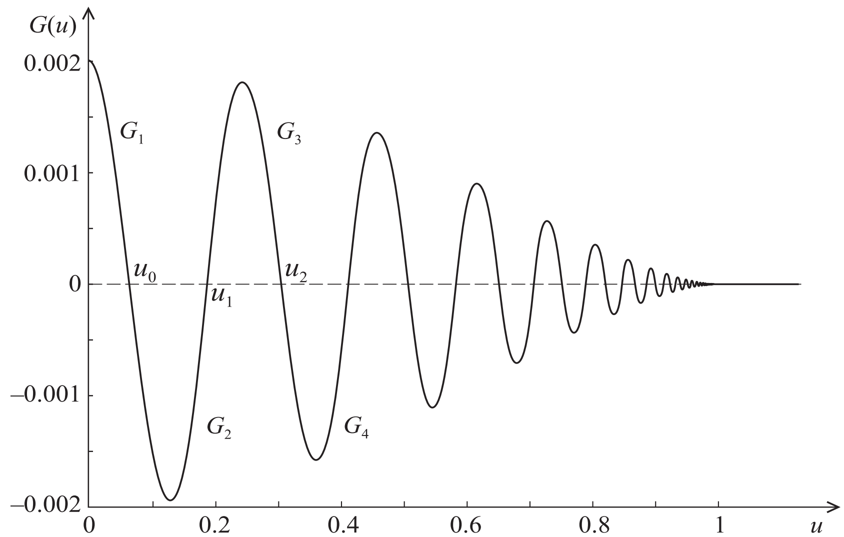

Figure 4: Example of numerical solution of the type III

4 Discussion

1. We have found the three families of self-similar solutions of the signum-Gordon equation in the $D = 3$ case. Comparing with the analogous results in 1+1 dimensions [4], we see certain similarity. In particular, in both cases there exist solutions of the type III (i.e., solutions vanishing for all $u \geq 1$) and of the type II (vanishing for all $u \geq u_1 > 1$). This seems to be a universal feature of the signum-Gordon equation. Solutions of the type I were not found in [4], but this may be a consequence of certain ad hoc restrictions on the form of solutions adopted there. On the other hand, in the $D = 3$ case we have not found spherically symmetric solutions that would correspond to the presented in [4] solutions with a negative velocity v. This can be related to the lack of translational invariance in the radial coordinate r.

2. The solutions presented above describe quite interesting processes, especially the type II solutions. The region $u \geq u_1 > 1$ in which $\varphi = 0$ corresponds to $r \leq t/u_1$. Thus, the field retreats completely from the region surrounding the origin. The radius of the spherical region with the vanishing field increases linearly with time. The corresponding radial velocity is equal to $1/u_1$, and it is smaller than 1. The presence of such subluminal velocities shows that actually the model should not be regarded as massless, in spite of the presence of the scale invariance. In truly massless models, such as the free massless scalar field

discussed in the next point, wave fronts always move with the velocity ± 1.

The type I solutions give $\varphi(r,t) = (1/6+\alpha_\infty)r^2 + 3\alpha_\infty t^2$ in the region $r \leq t$. Thus, in this case we have an accumulation of the field energy around the origin.

3. It is interesting to compare the solutions described in Section 3 above with self-similar radial solutions of the free wave equation which is obtained by dropping the signφ term from Eq. (1). It is clear that the general solution of the free wave equation is given by linear combination of the functions $g_1(u), g_2(u)$ introduced in Section 2. It turns out that solutions of the type II do not exist. Solutions of the type III have the form $G(u) = \alpha(1-u)^3$ for $u \leq 1$, $G(u) = 0$ for $u \geq 1$, where α is an arbitrary constant. On the plane $(G(0), G'(0))$ these solutions are represented by the points lying on the straight line $G'(0) = -3G(0)$, shown as the dashed line in Fig. 2. All other points in that plane would correspond in the case of the free wave equation to solutions of the type I, which have the following form: $G(u) = 2\beta(3u^2+1)$ for $u \geq 1$ and $G(u) = \beta(1+u)^3 + \alpha(1-u)^3$ for $u \leq 1$, where $\beta \neq 0$. Thus, the rather nontrivial shape of the region occupied by the solutions of the type III in Fig. 2, as well as the presence of the regions IIa, IIb, reflect the presence of the sign(φ) term.

5 Appendix. The proof of Lemma

1. If u_1 is the second order zero, the polynomial has the form (16) modulo the overall sign. The factor $2u_1u + u_1^2 - 3$ is strictly positive for all $u > u_1$ because $u_1 > 1$. Therefore G_d does not vanish for any $u > u_1$.

2. In the case u_1 is the first order zero, we have $G(u_1) = 0,\ G'(u_1) \neq 0$. Because of the symmetry $\varphi \to -\varphi$ we may consider only the case $G'(u_1) > 0$. Then, for $u \geq u_1$ we have the polynomial

$$G_+(u) = \frac{1}{6} - \frac{1+3u^2}{6(1+3u_1^2)} + \beta_+(u_1^3+3u_1)\left[\frac{u^3+3u}{u_1^3+3u_1} - \frac{1+3u^2}{1+3u_1^2}\right],$$

where

$$\beta_+ > \frac{u_1}{3(u_1^2-1)^2}$$

in order to ensure that $G'_+(u_1) > 0$. Direct calculation shows that $G''_+(u) > 0$ for all $u > u_1$. Therefore $G'_+(u) > 0$ for all $u > u_1$, and in consequence the values of $G_+(u)$ do not return to 0 in the region $u > u_1$.

References

[1] L. Debnath, *Nonlinear Partial Differential Equations*. Birkhäuser, Boston-Basel-Berlin, 2005. Section 8.11.

[2] G. I. Barenblatt, *Scaling, Self-similarity, and Intermediate Asymptotics*. Cambridge University Press, Cambridge-New York-Melbourne, 1996.

[3] P. Bizoń, Commun. Math. Phys. **215**, 45 (2000); P. Laurençot, Physica D **222**, 80 (2006); P. Bizoń, D. Maison and A. Wasserman, Nonlinearity **20**, 2061 (2007); M. Herrmann, B. Niethammer and J. J. L. Velaźquez, J. Diff. Eq. **247**, 2282 (2009); P. Biler, G. Karch and R. Monneau, Commun. Math. Phys. **294**, 145 (2010); C. B. Muratov, P. V. Gordon and S. Y. Shvartsman, Phys. Rev. E **84**, 041916 (2011).

[4] H. Arodź, P. Klimas and T. Tyranowski, Phys. Rev. E **73**, 046609 (2006); H. Arodź, P. Klimas and T. Tyranowski, Acta Phys. Pol. **B38**, 3099 (2007).

[5] H. Arodź, P. Klimas and T. Tyranowski, Acta Phys. Pol. **B38**, 2537 (2007).

[6] L. C. Evans, *Partial Differential Equations*. American Math. Society, Providence, RI, 1998.